Math 472 Homework Assignment 1

Problem 1.9.2. Let $p(x) = 1/2^x$, $x = 1, 2, 3, \ldots$, zero elsewhere, be the pmf of the random variable X. Find the mgf, the mean, and the variance of X.

Solution 1.9.2. Using the geometric series $a/(1-r) = \sum_{x=1}^{\infty} ar^{x-1}$ for $|r| < 1$, we are able to compute the mgf of X,

$$\begin{aligned} m(t) = E[e^{tX}] &= \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx}/2^x = \sum_{x=1}^{\infty} (e^t/2)^x \\ &= \frac{e^t/2}{1-(e^t/2)} = (2e^{-t}-1)^{-1}, \end{aligned}$$

for $t < \ln 2$. With $m(t) = (2e^{-t}-1)^{-1}$, we are able to compute the first and second derivatives of $m(t)$,

$$\begin{aligned} m'(t) &= 2e^{-t}(2e^{-t}-1)^{-2} \\ m''(t) &= 2e^{-t}(2e^{-t}+1)(2e^{-t}-1)^{-3}. \end{aligned}$$

The first and second moments of X are $\mu = m'(0) = 2$ and $\mu_2 = m''(0) = 6$, and the variance is $\sigma^2 = \mu_2 - \mu^2 = 6 - 4 = 2$. Therefore the mgf, the mean, and the variance of X are

$$m(t) = (2e^{-t}-1)^{-1}, \quad \mu = 2, \quad \sigma^2 = 2.$$

Problem 1.9.3. For each of the following distributions, compute

$$P(\mu - 2\sigma < X < \mu + 2\sigma).$$

(1) $f(x) = 6x(1-x), 0 < x < 1$, zero elsewhere.
(2) $p(x) = 1/2^x, x = 1, 2, 3, \ldots$, zero elsewhere.

Solution 1.9.3. (1) The mean and second moment are

$$\begin{aligned} \mu &= \int_0^1 xf(x)\,dx = \int_0^1 6x^2(1-x)\,dx = 1/2 \\ \mu_2 &= \int_0^1 x^2 f(x)\,dx = \int_0^1 6x^3(1-x)\,dx = 3/10, \end{aligned}$$

so the variance is $\sigma^2 = \mu_2 - \mu^2 = 3/10 - (1/2)^2 = 1/20$ and the standard deviation is $\sigma = 1/\sqrt{20} = \sqrt{5}/10 < 0.224$. Hence

$$\begin{aligned} P(\mu - 2\sigma < X < \mu + 2\sigma) &= P\left(\frac{1}{2} - \frac{\sqrt{5}}{5} < X < \frac{1}{2} + \frac{\sqrt{5}}{5}\right) \\ &= \int_{\frac{1}{2}-\frac{\sqrt{5}}{5}}^{\frac{1}{2}+\frac{\sqrt{5}}{5}} 6x(1-x)\,dx \\ &= \frac{11\sqrt{5}}{25} \approx 0.984. \end{aligned}$$

Remark: $f(x) = 6x(1-x)$ is the density for a Beta distribution with parameters $\alpha = 2, \beta = 2$, so you can quickly find the mean and variance using the equations on page 667.

(2) From problem 1.9.2, we know that $\mu = 2$ and $\sigma = \sqrt{2}$. Since $\mu - 2\sigma = 2 - 2\sqrt{2} < 0$ and $\mu + 2\sigma = 2 + 2\sqrt{2} \approx 4.82$

$$\begin{aligned} P(\mu - 2\sigma < X < \mu + 2\sigma) = P(X \leq 4) \\ = \sum_{x=1}^{4} \frac{1}{2^x} = \frac{15}{16} = 0.9375. \end{aligned}$$

Problem 1.9.5. Let a random variable X of the continuous type have a pdf $f(x)$ whose graph is symmetric with respect to $x = c$. If the mean value of X exists, show that $E[X] = c$.

Solution 1.9.5. Given that $f(c-x) = f(c+x)$, we will show that $E[X-c] = E[X] - c = 0$.

$$\begin{aligned} E[X-c] &= \int_{-\infty}^{\infty} (x-c) f(x)\, dx \\ &= \int_{-\infty}^{c} (x-c) f(x)\, dx + \int_{c}^{\infty} (x-c) f(x)\, dx. \end{aligned}$$

In the first integral, make the substitution $x = c - u, dx = -du$ and in the second integral make the substitution $x = c + u, dx = du$. Then

$$\begin{aligned} E[X-c] &= \int_{-\infty}^{c} (x-c) f(x)\, dx + \int_{c}^{\infty} (x-c) f(x)\, dx. \\ &= \int_{\infty}^{0} u f(c-u)\, du + \int_{0}^{\infty} u f(c+u)\, du \\ &= -\int_{0}^{\infty} u f(c+u)\, du + \int_{0}^{\infty} u f(c+u)\, du = 0, \end{aligned}$$

as desired. We conclude that if the density function for a random variable X is symmetric about the point c, then $\mu = E[X] = c$.

Problem 1.9.6. Let the random variable X have mean μ, standard deviation σ, and mgf $M(t)$, $-h < t < h$. Show that

$$E\left[\frac{X-\mu}{\sigma}\right] = 0, \qquad E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = 1, \quad \text{and}$$

$$E\left\{\exp\left[t\left(\frac{X-\mu}{\sigma}\right)\right]\right\} = e^{-\mu t/\sigma} M\left(\frac{t}{\sigma}\right), \qquad -h\sigma < t < h\sigma.$$

Solution 1.9.6. Using the linear properties of expected value (see Theorem 1.8.2) and the definition of $\mu = E[X]$, we calculate

$$E\left[\frac{X-\mu}{\sigma}\right] = \frac{E[X-\mu]}{\sigma} = \frac{E[X]-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0,$$

which verifies the first equation.

Using the linear properties of expected value again and the definition of $\sigma^2 = E[(X-\mu)^2]$, we calculate

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = E\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{E[(X-\mu)^2]}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1,$$

which verifies the second equation.

If $-h\sigma < t < h\sigma$ then $-h < t/\sigma < h$, which shows that t/σ is in the domain of M. Using the definition of $M(t) = E[\exp(tX)]$ and the linear properties of the expected value, we calculate

$$e^{-\frac{\mu t}{\sigma}} M\left(\frac{t}{\sigma}\right) = e^{-\frac{\mu t}{\sigma}} E\left[e^{\frac{t}{\sigma}X}\right] = E\left[e^{-\frac{\mu t}{\sigma}} e^{\frac{t}{\sigma}X}\right] = E\left[e^{\frac{t}{\sigma}X - \frac{\mu t}{\sigma}}\right] = E\left[e^{t\frac{(X-\mu)}{\sigma}}\right],$$

which verifies the third equation.

Problem 1.9.7. Show that the moment generating function of the random variable X having the pdf $f(x) = 1/3$, $-1 < x < 2$, zero elsewhere, is

$$M(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0. \end{cases}$$

Solution 1.9.7. As with every mgf, $M(0) = E[e^0] = E[1] = 1$. For $t \neq 0$,

$$M(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-1}^{2} \frac{e^{tx}}{3}\,dx = \left.\frac{e^{tx}}{3t}\right|_{-1}^{2} = \frac{e^{2t}-e^{-t}}{3t}.$$

Problem 1.9.11. Let X denote a random variable such that $K(t) = E[t^X]$ exists for all real values of t in a certain open interval that includes the point $t = 1$. Show that $K^{(m)}(1)$ is equal to the mth *factorial moment* $E[X(X-1)\cdots(X-m+1)]$.

Solution 1.9.11. Differentiating $k(t) = t^x$ m times we find

$$k^{(m)}(t) = x(x-1)\cdots(x-m+1)t^{x-m}.$$

We may therefore expand t^x in its Taylor series about $t = 1$

$$t^x = \sum_{m=0}^{\infty} k^{(m)}(1)\frac{(t-1)^m}{m!} = \sum_{m=0}^{\infty} x(x-1)\cdots(x-m+1)\frac{(t-1)^m}{m!}.$$

Using this Taylor series we see that

$$\begin{aligned} K(t) = E\left[t^X\right] &= E\left[\sum_{m=0}^{\infty} X(X-1)\cdots(X-m+1)\frac{(t-1)^m}{m!}\right] \\ &= \sum_{m=0}^{\infty} E\left[X(X-1)\cdots(X-m+1)\right]\frac{(t-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} K^{(m)}(1)\frac{(t-1)^m}{m!}. \end{aligned}$$

Comparing the last two series shows that

$$K^{(m)}(1) = E\left[X(X-1)\cdots(X-m+1)\right].$$

Problem 1.9.12. Let X be a random variable. If m is a positive integer, the expectation $E[(X-b)^m]$, if it exists, is called the mth moment of the distribution about the point b. Let the first, second, and third moments of the distribution about the point 7 be 3, 11, and 15, respectively. Determine the mean μ of X, and then find the first, second, and third moments of the distribution about the point μ.

Solution 1.9.12. We are given $E[X-7] = 3$, $E[(X-7)^2] = 11$, and $E[(X-7)^3] = 15$. Expanding the first equation gives

$$E[X-7] = E[X] - 7 = \mu - 7 = 3,$$

and therefore $\mu = 10$. Continuing the calculations,

$$\begin{aligned} E[(X-\mu)^2] &= E[(X-10)^2] = E\left\{[(X-7)-3]^2\right\} \\ &= E[(X-7)^2 - 6(X-7) + 9] = E[(X-7)^2] - 6E[X-7] + 9 \\ &= 11 - 18 + 9 = 2. \\ E[(X-\mu)^3] &= E[(X-10)^3] = E\left\{[(X-7)-3]^3\right\} \\ &= E[(X-7)^3] - 9E[(X-7)^2] + 27E[X-7] - 27 \\ &= 15 - 99 + 81 - 27 = -30. \end{aligned}$$

Thus the first, second, and third moments of X about the mean $\mu = 10$ are respectively 0, 2, and -30.

Problem 1.9.25. Let X be a random variable with a pdf $f(x)$ and mgf $M(t)$. Suppose f is symmetric about 0; i.e., $f(-x) = f(x)$. Show that $M(-t) = M(t)$.

Solution 1.9.25. We will use the substitution $x = -u, dx = -du$ in the following calculation.

$$\begin{aligned} M(-t) &= \int_{-\infty}^{\infty} e^{(-t)x} f(x)\,dx = \int_{-\infty}^{\infty} e^{t(-x)} f(x)\,dx \\ &= -\int_{\infty}^{-\infty} e^{tu} f(-u)\,du = \int_{-\infty}^{\infty} e^{tu} f(u)\,du \\ &= M(t). \end{aligned}$$

Problem 1.10.3. If X is a random variable such that $E[X] = 3$ and $E[X^2] = 13$, use Chebyshev's inequality to determine a lower bound for the probability $P(-2 < X < 8)$.

Solution 1.10.3. Chebyshev's inequality states that $P(|X - \mu| < k\sigma) \geq 1 - (1/k^2)$. In this problem $\mu = 3$ and $\sigma^2 = 13 - 9 = 4$, giving $\sigma = 2$. Thus

$$\begin{aligned} P(-2 < X < 8) &= P(-5 < X - 3 < 5) = P(|X - 3| < 5) \\ &= P(|X - 3| < \frac{5}{2}2) \\ &\geq 1 - \left(\frac{2}{5}\right)^2 = 1 - \frac{4}{25} = \frac{21}{25}. \end{aligned}$$

From the Chebyshev inequality we conclude that $P(-2 < X < 8) \geq 21/25$.

Problem 1.10.4. Let X be a random variable with mgf $M(t)$, $-h < t < h$. Prove that

$$P(X \geq a) \leq e^{-at} M(t), \qquad 0 < t < h,$$

and that

$$P(X \leq a) \leq e^{-at} M(t), \qquad -h < t < 0.$$

Solution 1.10.4. We will use Markov's inequality (Theorem 1.10.2), which states that if $u(X) \geq 0$ and if $c > 0$ is a constant, then

$$P(u(X) \geq c) \leq E[u(X)]/c.$$

We will also use the facts that *increasing* functions *preserve order* and *decreasing* functions *reverse order*.

Consider the function $u(x) = e^{tx} > 0$, which is an *increasing* function of x as long as $t > 0$. So, for $t > 0$ we have that $X \geq a$ if and only if $e^{tX} = u(X) \geq u(a) = e^{ta}$. Applying Markov's inequality and the definition of $M(t)$ we have for $0 < t < h$

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq E[e^{tX}]/e^{ta} = e^{-ta} M(t).$$

When $t < 0$, $u(x) = e^{tx} > 0$ is a *decreasing* function of x, so $X \leq a$ if and only if $e^{tX} = u(X) \geq u(a) = e^{ta}$. Applying Markov's inequality again we have for $-h < t < 0$

$$P(X \leq a) = P(e^{tX} \geq e^{ta}) \leq E[e^{tX}]/e^{ta} = e^{-ta} M(t).$$

Problem 1.10.5. The mgf of X exists for all real values of t and is given by

$$M(t) = \frac{e^t - e^{-t}}{2t}, t \neq 0, M(0) = 1.$$

Use the result of the preceding exercise to show that $P(X \geq 1) = 0$ and $P(X \leq -1) = 0$.

Solution 1.10.5. Taking $a = 1$ in problem 1.10.4, we see that for all $t > 0$, $P(X \geq 1) \leq e^{-t}M(t) = (1 - e^{-2t})/(2t)$. Taking the limit as $t \to \infty$

$$0 \leq P(X \geq 1) \leq \lim_{t\to\infty} \frac{1 - e^{-2t}}{2t} = 0$$

which shows that $P(X \geq 1) = 0$.

Taking $a = -1$ in problem 1.10.4, we see that for all $t < 0$, $P(X \leq -1) \leq e^t M(t) = (e^{2t} - 1)/(2t)$. Taking the limit as $t \to -\infty$

$$0 \leq P(X \leq -1) \leq \lim_{t\to-\infty} \frac{e^{2t} - 1}{2t} = 0$$

which shows that $P(X \leq -1) = 0$.

Problem 3.3.1. If $(1 - 2t)^{-6}$, $t < 1/2$, is the mgf of the random variable X, find $P(X < 5.23)$.

Solution 3.3.1. The mgf of X is that of a χ^2-distribution with $r = 12$ degrees of freedom. Using Table II on page 658, we see that the probability $P(X < 5.23) \approx 0.050$.

Problem 3.3.2. If X is $\chi^2(5)$, determine the constants c and d so that $P(c < X < d) = 0.95$ and $P(X < c) = 0.025$.

Solution 3.3.2. Using Table II on page 658 we find $P(X < 0.831) \approx 0.025$ and $P(X < 12.833) \approx 0.975$. So, with $c = 0.831$ and $d = 12.833$ we have $P(c < X < d) = 0.975 - 0.025 = 0.95$ and $P(X < c) = 0.025$.

Problem 3.3.3. Find $P(3.28 < X < 25.2)$ if X has a gamma distribution with $\alpha = 3$ and $\beta = 4$.

Solution 3.3.3. The mgf of X is $M_X(t) = (1 - 4t)^{-3}$. From this we see that $M(t/2) = E[e^{tX/2}] = (1 - 2t)^{-3}$, which is the mgf for a χ^2 with $r = 6$ degrees of freedom. Using Table II on page 658 we calculate

$$P(3.28 < X < 25.2) = P(1.64 < X/2 < 12.6) \approx 0.950 - 0.050 = 0.900.$$

Problem 3.3.5. Show that

$$\int_\mu^\infty \frac{z^{k-1}e^{-z}}{\Gamma(k)}\, dz = \sum_{x=0}^{k-1} \frac{\mu^x e^{-\mu}}{x!}, \quad k = 1, 2, 3, \ldots . \tag{1}$$

This demonstrates the relationship between the cdfs of the gamma and Poisson distributions.

Solution 3.3.5. An easy calculation shows that equation (1) is valid for $k = 1$, which establishes the base case for an induction proof.

The key to establishing the induction step is the following calculation:

$$\begin{aligned} \frac{\mu^k e^{-\mu}}{\Gamma(k+1)} &= -\frac{z^k e^{-z}}{\Gamma(k+1)}\Bigg|_{\mu}^{\infty} \\ &= \int_{\mu}^{\infty} \frac{d}{dz}\left[-\frac{z^k e^{-z}}{\Gamma(k+1)}\right] dz \\ &= \int_{\mu}^{\infty} -\frac{kz^{k-1}e^{-z}}{k\Gamma(k)} + \frac{z^k e^{-z}}{\Gamma(k+1)} dz \\ &= \int_{\mu}^{\infty} -\frac{z^{k-1}e^{-z}}{\Gamma(k)} + \frac{z^k e^{-z}}{\Gamma(k+1)} dz. \end{aligned} \tag{2}$$

Now add $\mu^k e^{-\mu}/\Gamma(k+1)$ to both sides of equation (1) to obtain

$$\int_{\mu}^{\infty} \frac{z^{k-1}e^{-z}}{\Gamma(k)}\, dz + \frac{\mu^k e^{-\mu}}{\Gamma(k+1)} = \sum_{x=0}^{k-1} \frac{\mu^x e^{-\mu}}{x!} + \frac{\mu^k e^{-\mu}}{\Gamma(k+1)} = \sum_{x=0}^{k} \frac{\mu^x e^{-\mu}}{x!}.$$

Using equation (2) to simplify the left hand side of the previous equation yields

$$\int_{\mu}^{\infty} \frac{z^k e^{-z}}{\Gamma(k+1)}\, dz = \sum_{x=0}^{k} \frac{\mu^x e^{-\mu}}{x!},$$

which shows that equation (1) is true for $k+1$ if it is true for k. Therefore, by the principle of mathematical induction, equation (1) is true for all $k = 1, 2, 3, \ldots$.

Problem 3.3.9. Let X have a gamma distribution with parameters α and β. Show that $P(X \geq 2\alpha\beta) \leq (2/e)^{\alpha}$.

Solution 3.3.9. From appendix D on page 667, we see that the mgf for X is $M(t) = (1 - \beta t)^{-\alpha}$ for $t < 1/\beta$. Problem 1.10.4 shows that for every constant a and for every $t > 0$ in the domain of $M(t)$, $P(X \geq a) \leq e^{-at}M(t)$. Applying this result to our gamma distributed random variable we have for all $0 < t < 1/\beta$

$$P(X \geq 2\alpha\beta) \leq \frac{e^{-2\alpha\beta t}}{(1-\beta t)^{\alpha}}.$$

Let us try to find the minimum value of $y = e^{-2\alpha\beta t}(1-\beta t)^{-\alpha}$ over the interval $0 < t < 1/\beta$. A short calculation shows that the first two derivatives of y are

$$\begin{aligned} y' &= \alpha\beta e^{-2\alpha\beta}(1-\beta t)^{-\alpha-1}(2\beta t - 1) \\ y'' &= \alpha\beta^2 e^{-2\alpha\beta}(1-\beta t)^{-\alpha-2}\left[1 + \alpha(2\beta t - 1)^2\right]. \end{aligned}$$

Since $y' = 0$ at $t = 1/(2\beta) \in (0, 1/\beta)$ and since $y'' > 0$ we see that y takes its minimum value at $t = 1/(2\beta)$ and therefore

$$P(X \geq 2\alpha\beta) \leq \left.\frac{e^{-2\alpha\beta t}}{(1-\beta t)^{\alpha}}\right|_{t=\frac{1}{2\beta}} = \left(\frac{2}{e}\right)^{\alpha}.$$

Problem 3.3.15. Let X have a Poisson distribution with parameter m. If m is an experimental value of a random variable having a gamma distribution with $\alpha = 2$ and $\beta = 1$, compute $P(X = 0, 1, 2)$.

Solution 3.3.15. We will be using techniques from the topic of joint and conditional distributions. We are given that m has a gamma distribution with $\alpha = 2$ and $\beta = 1$, therefore its *marginal* probability density function is $f_m(m) = me^{-m}$ for $m > 0$, zero elsewhere. For the random variable X, we are given the *conditional* probability mass function given m is $p(x \mid m) = m^x e^{-m}/x!$ for $x = 0, 1, 2, \ldots$ and $m > 0$, zero elsewhere. From the given information we are able to determine that the *joint* mass-density function is

$$f(x, m) = p(x \mid m) f_m(m) = m^{x+1} e^{-2m}/x!$$

for $x = 0, 1, 2, \ldots$ and $m > 0$, zero elsewhere. We calculate the *marginal* probability mass function for X,

$$\begin{aligned}
p_X(x) &= \int_{-\infty}^{\infty} f(x, m)\, dm \\
&= \int_0^{\infty} \frac{m^{x+1} e^{-2m}}{x!}\, dm \qquad \text{(letting } u = 2m,\ du = 2dm\text{)} \\
&= \frac{1}{2^{x+2} x!} \int_0^{\infty} u^{(x+2)-1} e^{-u}\, du \\
&= \frac{\Gamma(x+2)}{2^{x+2} x!} \\
&= \frac{x+1}{2^{x+2}},
\end{aligned}$$

for $x = 0, 1, 2, \ldots$, zero elsewhere. This allows us to find

$$\begin{aligned}
P(X = 0) &= p_X(0) = \frac{1}{4}, \\
P(X = 1) &= p_X(1) = \frac{1}{4}, \\
P(X = 2) &= p_X(2) = \frac{3}{16}.
\end{aligned}$$

3.3.19. Determine the constant c in each of the following so that each $f(x)$ is a β pdf:

(1) $f(x) = cx(1-x)^3$, $0 < x < 1$, zero elsewhere.
(2) $f(x) = cx^4(1-x)^5$, $0 < x < 1$, zero elsewhere.
(3) $f(x) = cx^2(1-x)^8$, $0 < x < 1$, zero elsewhere.

Solution 3.3.19.

(1) $c = 1/B(2,4) = \Gamma(6)/(\Gamma(2)\Gamma(4)) = 5!/(1!3!) = 20.$
(2) $c = 1/B(5,6) = \Gamma(11)/(\Gamma(5)\Gamma(6)) = 10!/(4!5!) = 1260.$
(3) $c = 1/B(3,9) = \Gamma(12)/(\Gamma(3)\Gamma(9)) = 11!/(2!8!) = 495.$

Problem 3.3.22. Show, for $k = 1, 2, \ldots, n$, that

$$\int_p^1 \frac{n!}{(k-1)!(n-k)!} z^{k-1}(1-z)^{n-k} dz = \sum_{x=0}^{k-1} \binom{n}{x} p^x (1-p)^{n-x}.$$

This demonstrates the relationship between the cdfs of the β and binomial distributions.

Solution 3.3.22. This problem is very similar to problem 3.3.5. In this case the key to the induction step is the following calculation:

$$\begin{aligned}
\binom{n}{k} p^k (1-p)^{n-k} &= -\frac{n!}{k!(n-k)!} z^k (1-z)^{n-k} \Big|_p^1 \\
&= \int_p^1 \frac{d}{dz}\left[-\frac{n!}{k!(n-k)!} z^k (1-z)^{n-k}\right] dz \\
&= \int_p^1 -\frac{kn!}{k!(n-k)!} z^{k-1}(1-z)^{n-k} + \frac{(n-k)n!}{k!(n-k)!} z^k (1-z)^{n-k-1}\, dz \\
&= \int_p^1 -\frac{n!}{(k-1)!(n-k)!} z^{k-1}(1-z)^{n-k} + \frac{n!}{k!(n-k-1)!} z^k (1-z)^{n-k-1}\, dz
\end{aligned}$$

The rest of the proof is similar to the argument in problem 3.3.5.

Remark: an alternate proof is to observe that $\sum_{x=0}^{k-1} \binom{n}{x} p^x (1-p)^{n-x}$ is a telescoping series, when you take advantage of the above calculation.

Problem 3.3.23. Let X_1 and X_2 be independent random variables. Let X_1 and $Y = X_1 + X_2$ have chi-square distributions with r_1 and r degrees of freedom, respectively. Here $r_1 < r$. Show that X_2 has a chi-square distribution with $r - r_1$ degrees of freedom.

Solution 3.3.23. From appendix D on page 667, we see that the mgfs of X_1 and Y are, respectively, $M_{X_1}(t) = (1-2t)^{-r_1/2}$ and $M_Y(t) = (1-2t)^{-r/2}$. Since $Y = X_1 + X_2$ is the sum of independent random variables, $M_Y(t) = M_{X_1}(t) M_{X_2}(t)$ (by Theorem 2.2.5). Solving, we find that

$$M_{X_2}(t) = \frac{M_Y(t)}{M_{X_1}(t)} = \frac{(1-2t)^{-r/2}}{(1-2t)^{-r_1/2}} = (1-2t)^{-(r-r_1)/2},$$

which is the mgf of a chi-square random variable with $r - r_1$ degrees of freedom. Therefore, by Theorem 1.9.1, X_2 has a chi-square distribution with $r - r_1$ degrees of freedom.

Problem 3.3.24. Let X_1, X_2 be two independent random variables having gamma distributions with parameters $\alpha_1 = 3$, $\beta_1 = 3$ and $\alpha_2 = 5$, $\beta_2 = 1$, respectively.

(1) Find the mgf of $Y = 2X_1 + 6X_2$.
(2) What is the distribution of Y?

Solution 3.3.24. (1) The mgfs of X_1 and X_2 are, respectively,

$$M_{X_1}(t) = (1-3t)^{-3} \text{ and } M_{X_2}(t) = (1-t)^{-5}.$$

Since X_1 and X_2 are independent, theorem 2.5.4 implies that

$$\begin{aligned}
M_Y(t) &= E\left[e^{t(2X_1+6X_2)}\right] \\
&= E\left[e^{2tX_1}e^{6tX_2}\right] \\
&= E\left[e^{2tX_1}\right]E\left[e^{6tX_2}\right] \\
&= M_{X_1}(2t)M_{X_2}(6t) \\
&= (1-3(2t))^{-3}(1-1(6t))^{-5} \\
&= (1-6t)^{-8}.
\end{aligned}$$

(2) Since $(1-6t)^{-8}$ is the mgf of a gamma distribution with parameters $\alpha = 8$ and $\beta = 6$, Theorem 1.9.1 shows that Y has a gamma distribution with parameters $\alpha = 8$ and $\beta = 6$.